THE INFLUENCE OF NON-UNIFORM HEATING ON THE STABILITY OF A COMPRESSED BAR

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Considered is the stability of a clamped, compressed bar of a rectangular cross-section undergoing small elastic-plastic deformations under conditions of non-uniform heating, for an arbitrary compression diagram of the material. The temperature field $T(x,y,z)$ is assumed constant along the length of the bar and varying linearly across the cross-section in the direction of buckling. For assumed boundary conditions the non-uniform heating does not cause the curving of the axis of the bar. The problem is reduced to finding the critical force from a homogeneous differential equation.

For the case of an ideally plastic material and for a material with a linear strain hardening, results can be obtained in a closed form. The formulation of this problem is due to L.I. Balabukh.

Suppose a clamped bar of a constant rectangular cross-section is non-uniformly heated by a steady temperature field

$$T(x, y, z) = \frac{t_2 + t_1}{2} - \frac{t_2 - t_1}{h} y, \qquad t_2 > t_1 \tag{1}$$

and is acted upon by a compressive force. Here, the x-axis coincides with the axis of the bar, y and z are principal central axes of the cross-section, t_1 and t_2 are temperatures in the boundary fibers, measured from some initial temperature T_0, h_0 is the height of the cross-section.

Assume the total shortening of the bar to be u. The compressive stresses causing this shortening are

$$\sigma = \sigma\left(\frac{u}{l} + \alpha T\right) \tag{2}$$

where l is the length of the bar. We introduce the notations

$$\frac{u}{l} = \varepsilon^\circ, \qquad \frac{u}{l} + \alpha T = \varepsilon \tag{3}$$

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The coefficient of linear expansion α is assumed to be temperature independent. The compression diagram of the material for an initial temperature is assumed to be quite arbitrary.

In order to simplify the solution of the problem at hand we assume the following simple dependence of the stress-strain diagram, $\sigma - \epsilon$, on the temperature T:

$$\frac{\sigma(\varepsilon, T)}{\sigma_0(\varepsilon, T_0)} = 1 - \frac{1}{E_0}\frac{dE}{dT}(T - T_0) \tag{4}$$

This dependence approximates with sufficient accuracy compression curves of such materials as high-strength steels and aluminum alloys.

Consider a small deviation $f(x)$ of the bar axis from a straight line. Infinitesimally small increments of bending stress in zones of additional loading and unloading are

$$\delta\sigma = \begin{cases} \dfrac{d\sigma}{d\varepsilon}\delta\varepsilon = \dfrac{d\sigma}{d\varepsilon}(y - y_0)\varkappa, & \delta\varepsilon > 0 \\ E\delta\varepsilon = E(y - y_0)\varkappa, & \delta\varepsilon < 0 \end{cases} \tag{5}$$

where $\kappa = -f''(x)$ is the curvature of the elastic line of the bar, y_0 is the coordinate, determining the position of the neutral line of the cross-section.

Assuming that the buckling of the bar occurs for a constant value of the axial force, we obtain

$$J_1(\varepsilon^\circ, \overline{y_0}) + \overline{J_2}(\varepsilon^\circ, \overline{y_0}) = 0 \qquad \left(J_1 = \int_{y_1}^{0} \frac{d\sigma}{d\varepsilon}(y - y_0)\,dy, \quad J_2 = \int_{0}^{y_2} E(y - y_0)\,dy\right) \tag{6}$$

Here

$$y_1 = -\left(\tfrac{1}{2}h + y_0\right), \qquad y_2 = \left(\tfrac{1}{2}h - y_0\right), \ \overline{y_0} = 2\frac{y_0}{h}$$

The condition (6), together with relations (1) to (4), enables determination of the position of the neutral line for an arbitrary diagram of $\sigma_0 = \sigma_0(\epsilon)$.

By determining the deformation ϵ as a function of coordinate y according to (1) and (2) and performing a substitution of variables

$$y = \frac{h}{\alpha(t_2 - t_1)}\left[-\varepsilon + \varepsilon^\circ + \frac{\alpha(t_2 + t_1)}{2}\right], \qquad T = \frac{1}{\alpha}(\varepsilon - \varepsilon^\circ) \tag{7}$$

in expression for $J_1(\epsilon^0, \overline{y}_0)$ we obtain

$$J_1 = \left[\frac{h}{\alpha(t_2 - t_1)}\right]^2 \int_{\varepsilon_1}^{\varepsilon_2} \frac{d\sigma_0(\varepsilon)}{d\varepsilon}\left[1 - \frac{1}{\alpha E_0}\frac{dE}{dT}(\varepsilon - \varepsilon^\circ - \alpha T_0)\right](\varepsilon - \varepsilon_2)\,d\varepsilon \tag{8}$$

where

$$\varepsilon_1 = \varepsilon^\circ + \alpha t_2, \qquad \varepsilon_2 = \varepsilon^\circ + \frac{\alpha}{2}\left[(t_2 + t_1) - (t_2 - t_1)\overline{y_0}\right]$$

Considering expression (3), the integral $J_2(\epsilon^0, \overline{y}_0)$ may be expressed as

$$J_2 = \frac{h^2}{8}\left\{1 + \frac{T_0}{E_0}\frac{dE}{dT} - \frac{1}{2E_0}\frac{dE}{dT}\left[(t_2 + t_1) - (t_2 - t_1)\bar{y}_0\right]\right\}(1 - \bar{y}_0)^2 + \\ + \frac{h^2}{24E_0}\frac{dE}{dT}(t_2 - t_1)(1 - \bar{y}_0)^3 \qquad (9)$$

The solution of equation

$$J_1(\varepsilon^\circ, \bar{y}_0) + J_2(\varepsilon^\circ, \bar{y}_0) = \Phi_1(\varepsilon^\circ, \bar{y}_0) = 0$$

with respect to $\bar{y}_0$, for various values of the parameter ϵ^0, can be obtained for the general case of the diagram $\sigma_0 = \sigma_0(\epsilon)$ by means of numerical methods.

The variation of the bending moment during the buckling of the bar is

$$\delta M = -\iint \delta\sigma y\, dy\, dz = -\varkappa\left[b\int_{y_1}^{0}\frac{d\sigma}{d\varepsilon}(y - y_0)^2 dy + b\int_{0}^{y_2} E(y - y_0)^2 dy\right] = -KJ\varkappa \qquad (10)$$

where K is the reduced modulus, analogous in its content to Engesser-Karman modulus, b is the width of the rectangular cross-section:

Therefrom

$$K = J_3(\varepsilon^\circ, \bar{y}_0) + J_4(\varepsilon^\circ, \bar{y}_0) = \Phi_2(\varepsilon^\circ, \bar{y}_0) \qquad (11)$$

Here

$$J_3 = \frac{12}{h^3}\int_{y_1}^{0}\frac{d\sigma}{d\varepsilon}(y - y_0)^2 dy, \qquad J_4 = \frac{12}{h^3}\int_{0}^{y_2} E(y - y_0)^2 dy$$

Changing variables in accordance with (7) we obtain

$$J_3 = -\frac{12}{\alpha^3(t_2 - t_1)^3}\int_{\varepsilon_1}^{\varepsilon_2}\frac{d\sigma_0(\varepsilon)}{d\varepsilon}\left[1 - \frac{1}{\alpha E_0}\frac{dE}{dT}(\varepsilon - \varepsilon^\circ - \alpha T_0)\right](\varepsilon - \varepsilon_2)^2 d\varepsilon \qquad (12)$$

$J_4(\epsilon^0, \bar{y}_0)$ may be found by direct integration:

$$J_4 = \frac{1}{2}\left\{1 + \frac{T_0}{E_0}\frac{dE}{dT} - \frac{1}{2E_0}\frac{dE}{dT}\left[(t_2 + t_1) - (t_2 - t_1)\bar{y}_0\right]\right\}(1 - \bar{y}_0)^3 + \\ + \frac{3}{16E_0}\frac{dE}{dT}(t_2 - t_1)(1 - \bar{y}_0)^4 \qquad (13)$$

The compressive force N in the bar and the corresponding mean stress σ_N are:

$$N = \iint \sigma(\varepsilon)\, dy\, dz, \qquad \sigma_N = \frac{1}{h}\int_{-1/2 h}^{+1/2 h}\sigma(\varepsilon)\, dy \qquad (14)$$

$$\sigma_N(\varepsilon^\circ) = \frac{1}{\alpha(t_2 - t_1)}\left[1 + \frac{1}{\alpha E_0}\frac{dE}{dT}(\varepsilon^\circ + \alpha T_0)\right]\int_{\varepsilon_2}^{\varepsilon_1}\sigma_0(\varepsilon)\, d\varepsilon -$$

or

$$-\frac{1}{\alpha^2 (t_2 - t_1) E_0} \frac{dE}{dT} \int^{\varepsilon_1} \sigma_0 (\varepsilon) \varepsilon d\varepsilon \qquad (\varepsilon_3 = \varepsilon^\circ + \alpha t_1) \tag{15}$$

The critical value of the compressive force is †.

$$N_k = KJ \int_0^l f''^2 dx \Big/ \int_0^l f'^2 dx \tag{16}$$

For boundary conditions here considered

$$f(x) = f_0 \sin^2 \frac{\pi x}{l} \tag{17}$$

The critical slenderness is

$$\lambda(\varepsilon^\circ) = 2\pi \sqrt{\frac{K(\varepsilon^\circ)}{\sigma_N(\varepsilon^\circ)}} \tag{18}$$

Eliminating ϵ^0 from (16) and (18), we find the stability limit in the form

$$\sigma_k = \Phi(\lambda) \tag{19}$$

Here σ_k is the critical value of the mean compressive stress in the bar.

Consider further a special case $t_1 = T_0 = 0$, $t_2 = t$. Separating in the expression for $\sigma_0(\epsilon)$ the elastic part, we will assume the compression diagram of material at initial temperature in the form

$$\sigma_0(\varepsilon) = E_0 \varepsilon - \varphi_0(\varepsilon), \quad \text{where} \begin{cases} \varphi_0 = 0 & (\varepsilon \leqslant \varepsilon_p) \\ \varphi_0 = \varphi_0(\varepsilon) & (\varepsilon \geqslant \varepsilon_p) \end{cases} \tag{20}$$

and ϵ_p is the strain corresponding to the limit of elasticity.

Equations (6), (11) and (15) are simplified thereby. Performing calculations we obtain the following equation determining the coordinate y_0:

$$\frac{E_0}{6} [B - 3(1 - B)\bar{y}_0] - \frac{(2B - 1)}{2\alpha t}(1 + \bar{y}_0)\varphi_0(\varepsilon_1) = 2C_1 \int_{\varepsilon_1}^{\varepsilon_2} \varphi_0(\varepsilon)\varepsilon d\varepsilon - C_2 \int_{\varepsilon_1}^{\varepsilon_2} \varphi_0(\varepsilon) d\varepsilon \tag{21}$$

Here

$$C_1 = \frac{1}{(\alpha t)^2} A, \qquad A = \frac{1}{\alpha E_0} \frac{dE}{dT}$$

$$C_2 = \frac{1}{(\alpha t)^2} [1 + 2A\varepsilon^\circ + B(1 - \bar{y}_0)], \qquad B = \frac{t}{2E_0} \frac{dE}{dT}$$

† The reduced modulus K is determined from its smallest value corresponding to the buckling of the bar in direction of the less heated fiber.

The expression for the reduced modulus and the mean compressive stress† is

$$K = E_0\left[1 - B(1+2\overline{y_0}) + 3(1 - B)\overline{y_0}^2\right] - \frac{3(1-2B)}{at}(1+\overline{y_0})^2\varphi_0(\varepsilon_1) + + 3D_1\int_{\varepsilon_1}^{\varepsilon_2}\varphi_0(\varepsilon)\varepsilon^2 d\varepsilon - 2D_2\int_{\varepsilon_1}^{\varepsilon_2}\varphi_0(\varepsilon)\varepsilon\, d\varepsilon + D_3\int_{\varepsilon_1}^{\varepsilon_2}\varphi_0(\varepsilon)\, d\varepsilon \tag{22}$$

$$\sigma_N = E_0\left[(1-B)\varepsilon^\circ + \left(\frac{1}{2} - \frac{2}{3}B\right)at\right] - \frac{1}{at}(1 + A\varepsilon^\circ)\int_{\varepsilon^\circ}^{\varepsilon_1}\varphi_0(\varepsilon)\, d\varepsilon + \frac{A}{at}\int_{\varepsilon^\circ}^{\varepsilon_1}\varphi_0(\varepsilon)\varepsilon\, d\varepsilon \tag{23}$$

Coefficients D_1, D_2 and D_3 are calculated from the following formulas:

$$D_1 = \frac{12}{(at)^3}A$$

$$D_2 = \frac{12}{(at)^3}\left[1 + 3A\varepsilon^\circ + 2B(1 - \overline{y_0})\right]$$

$$D_3 = \frac{12}{(at)^3}\left[3A(\varepsilon^\circ)^2 + 2\varepsilon^\circ + (at + 4B\varepsilon^\circ)(1 - \overline{y_0}) + \frac{Bat}{2}(1 - \overline{y_0})^2\right]$$

For a material with a linear strain hardening

$$\varphi_0(\varepsilon) = 0, \qquad \varepsilon \leqslant \varepsilon_s; \qquad \varphi_0(\varepsilon) = (E_0 - E_0')(\varepsilon - \varepsilon_s), \qquad \varepsilon \geqslant \varepsilon_s \tag{24}$$

where ϵ_s is the deformation corresponding on the diagram $\sigma_0 = \sigma_0(\epsilon)$ to the point of transition into the region of plastic deformations with the modulus of hardening E_0'.

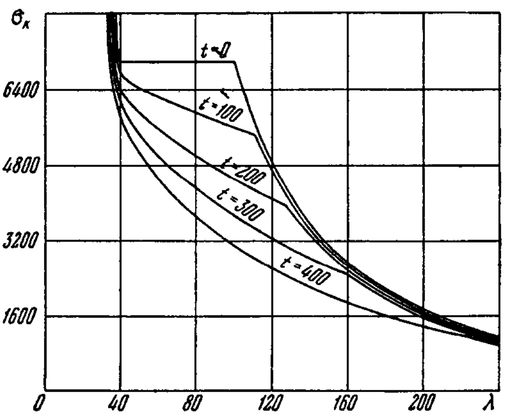

Fig. 1.

† The numerical determination of the reduced modulus should be done with great accuracy, since the last four terms of relation (22) differ little in their absolute values.

In this case, in evaluating the integrals appearing in equations (21), (22) and (23), one has to remember that for $\varepsilon \leqslant \varepsilon_s = \varepsilon^\circ + 1/2\, \alpha t (1 - \bar{y}_s)$ expressions under the integral become zero.

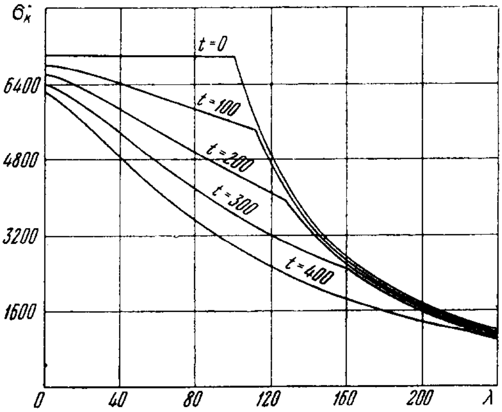

Fig. 2.

The value $\bar{y}_s = 2\, \bar{y}_s / h$ corresponds to the limit between the elastic and plastic zones of cross-sections for given values of ϵ_s, t and ϵ^0.

Expressions for the reduced modulus and for the stress σ_N may be obtained in a closed form:

$$K = \frac{E_0}{2} [1 - B(1 - \bar{y}_0)] \{[(1 - \bar{y}_0)^3 + (\bar{y}_0 - \bar{y}_s)^3] + \bar{E}_0 [(1 + \bar{y}_0)^3 - (\bar{y}_0 - \bar{y}_s)^3]\} + \frac{3BE_0}{8} \{[(1 - \bar{y}_0)^4 - (\bar{y}_0 - \bar{y}_s)^4] - \bar{E}_0 [(1 + \bar{y}_0)^4 - (\bar{y}_0 - \bar{y}_s)^4]\} \tag{25}$$

$$\sigma_N = E_0 \left[(1 - B)\varepsilon^\circ + \left(\frac{1}{2} - \frac{2}{3} B\right) \alpha t\right] - (E_0 - E_0') \left[\frac{B\alpha t}{8} \bar{y}_s + \frac{\alpha t}{4} \left(\frac{1}{2} - B\right) (1 + \bar{y}_s)^2 + \frac{B\alpha t}{24} (1 + \bar{y}_s)^3\right] \tag{26}$$

where $\bar{y}_0$ is determined from the equation

$$1 + \bar{E}_0 + (1 - \bar{E}_0)[H(1 - \bar{y}_s^2) - \bar{y}_s^3] - \{2H[1 - \bar{y}_s + \bar{E}_0(1 + \bar{y}_s)] + \frac{3}{2}(1 - \bar{E}_0)(1 - \bar{y}_s^2)\} \bar{y}_0 = 0 \tag{27}$$

Here

$$\bar{E}_0 = E_0' / E_0, \qquad H = 3(1 - B) / 2B$$

For $\bar{y}_0 \leqslant \bar{y}_s$ the plastic zone coincides with the zone of additional loading, hence in equation (27) one should set $\bar{y}_s = \bar{y}_0$. In this case we have a third order equation with respect to $\bar{y}_0$

$$1 + \bar{E}_0 - \frac{1}{2}(1 - \bar{E}_0)(3\bar{y}_0 - \bar{y}_0^3) + H[(1 - \bar{E}_0)(1 + \bar{y}_0^2) - 2(1 + \bar{E}_0)\bar{y}_0] = 0 \tag{28}$$

By putting in equations (25) to (28) $E_0' = E_0 = 0$, we obtain the solution of this problem for the case of a material with a yield point.

Figs. 1 and 2 show results of calculations of the limit of stability for a material with a linear strain hardening, with $E_0 = 1.75 \times 10^6$ kg/cm^2, $E_0' = 0.111 \times 10^6$ kg/cm^2, $\epsilon_s = 0.0040$, $dE/dT = 10^3$ kg/cm^{2}deg., $\alpha = 16 \times 10^{-6}$, and for an ideally plastic material having the same values of E_0, ϵ_s, dE/dT and α, respectively.

The influence of heating is particularly pronounced for the case of small elastic-plastic deformations and for medium values of the bar slenderness.

Translated by B.Z.